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TOROIDAL SHELLS UNDER NONSYMMETRIC LOADING

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Abstract—The complete asymptotic expansions of four homogeneous solutions, and a particular solution of toroidal shells, based on Novozhilov's thin shell equations, are given, which are valid for the stress and deformation of toroidal shells of circular cross section subjected to nonsymmetric loadings.

NOTATION

h	wall thickness of shells (constant)
r_0	radius of the meridian circle
R_0	distance from top of toroid to axis of revolution
μ	r_0/R_0
m	Fourier harmonic index
E	elastic modulus
v	Poisson ratio
λ^2	$i\sqrt{12(1-v^2)}\mu\dfrac{r_0}{h}m^4$
ε	$h/\sqrt{12(1-v^2)}$
θ, φ	tangential and circumferential angles of shell
σ	$1+\mu\sin\theta$
ξ_1, ξ_2	principal coordinates
A_1, A_2	Lame's coefficients
R_1, R_2	principal radii of curvature
q_1, q_2, q_n	tangential, circumferential and normal loading components
N_1, N_2, N_{12}, N_{21}	stress resultants, shown in Fig. 1
M_1, M_2, M_{12}, M_{21}	stress couples, shown in Fig. 1
S	$N_{12}-\dfrac{M_{21}}{R_2}=N_{21}-\dfrac{M_{12}}{R_1}$
H	$\frac{1}{2}(M_{12}+M_{21})$
κ_1, κ_2, τ	change in curvature and change in twist
$\tilde{N}_1$	$N_1-i\varepsilon Eh\kappa_2$
$\tilde{N}_2$	$N_2-i\varepsilon Eh\kappa_1$
$\tilde{S}$	$S+i\varepsilon Eh\tau$
$\tilde{T}$	$\tilde{N}_1+\tilde{N}_2$
u, v, w	tangential, circumferential and normal displacements, respectively.

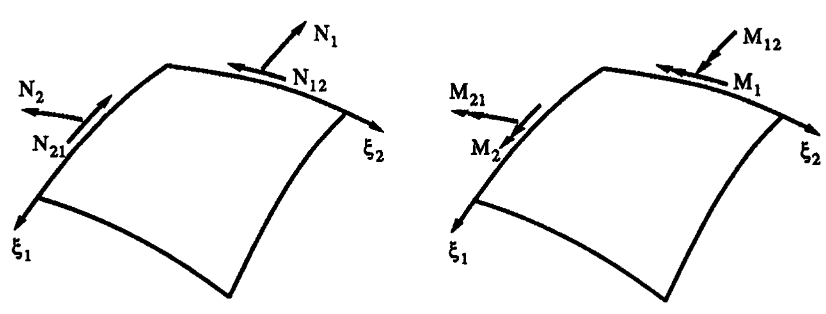

Fig. 1. Notation of force resultants and moment resultants in shell coordinates.

1. INTRODUCTION

The toroidal shell is characterized by the existence of the transition points at $\theta = 0$ and π. Therefore, the displacement at this point is discontinuous in the membrane theory and bending moments always exist. This feature makes it difficult to obtain the asymptotic solutions.

The toroidal shell problems are generally divided into two cases to be investigated according to the loadings; axisymmetric ($m = 0$) or nonsymmetric ($m \neq 0$). In the present paper, only the latter is discussed.

The case of axisymmetric loadings has been investigated for many years. However, there are few papers on the case of nonsymmetric loadings. Steele (1959) solved this problem for the first time in his dissertation at Stanford University. He derived the nonhomogeneous integro-differential equation of the fourth order and obtained an asymptotic solution for large values of $\mu = \sqrt{12(1-v^2)r_0^2/(R_0h)}$ and for $m^3/(\sqrt{12}(r_0/h)) \ll 1$. The latter condition restricted his solution adaptable for only lower harmonics $m = 0, 1, 2, \ldots$

In the present investigation, it is identified that the comparison equation is, in fact, the certain generalization of Airy's equation. Therefore, the generalized Airy functions, introduced by Drazin and Reid (1981), are used to obtain successfully the complete asymptotic expansions of all four homogeneous solutions and a particular solution. They are numerically satisfactory, uniformly valid and they satisfy the accuracy of the theory of thin shells.

As for the case of axisymmetric loading, a novel solution has been found by the author. The paper will be dispatched separately. The goal of the author is to obtain a unified solution for the toroidal shells under arbitrary loadings.

The solution of toroidal shells with nonsymmetric loading corresponds to a transition point problem for differential equations of fourth order. The transition point of higher order occurs in shell vibrations. Zhang and Zhang (1991) have obtained the complete uniformly valid solutions.

2. FUNDAMENTAL EQUATIONS

Novozhilov (1951) has given the thin shell equations in the complex form

$$\frac{\partial A_2\tilde{N}_1}{\partial \xi_1} + \frac{1}{A_1}\frac{\partial A_1^2\tilde{S}}{\partial \xi_2} - \frac{\partial A_2}{\partial \xi_1}\tilde{N}_2 + i\varepsilon\frac{A_2}{R_1}\frac{\partial \tilde{T}}{\partial \xi_1} = -A_1A_2q_1$$

$$\frac{\partial A_1\tilde{N}_2}{\partial \xi_2} + \frac{1}{A_2}\frac{\partial A_2^2\tilde{S}}{\partial \xi_1} - \frac{\partial A_1}{\partial \xi_2}\tilde{N}_1 + i\varepsilon\frac{A_1}{R_2}\frac{\partial \tilde{T}}{\partial \xi_2} = -A_1A_2q_2$$

$$\frac{\tilde{N}_1}{R_1} + \frac{\tilde{N}_2}{R_2} - i\varepsilon\frac{1}{A_1A_2}\left\{\frac{\partial}{\partial \xi_1}\left(\frac{A_2}{A_1}\frac{\partial \tilde{T}}{\partial \xi_1}\right) + \frac{\partial}{\partial \xi_2}\left(\frac{A_1}{A_2}\frac{\partial \tilde{T}}{\partial \xi_2}\right)\right\} = q_n. \tag{1}$$

For the case of the toroidal shells of circular cross section, as shown in Fig. 2, the parameters are now

$$\xi_1 = \theta, \quad \xi_2 = \varphi; \qquad A_1 = r_0, \quad A_2 = R_0\sigma; \qquad R_1 = r_0, \quad R_2 = R_0\sigma/\sin\theta. \tag{2}$$

Furthermore, with the introduction of a new complex variable

$$\tilde{U} = R_0\sigma\sin\theta\tilde{N}_1 - i\varepsilon\frac{R_0\sigma}{r_0}\cos\theta\frac{\partial \tilde{T}}{\partial \theta}, \tag{3}$$

eqn (1) can be rewritten as

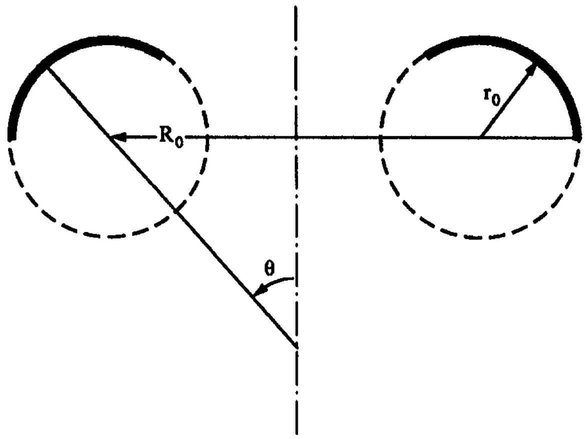

Fig. 2. Toroidal shells of circular cross section.

$$\frac{R_0}{r_0^2\sigma}\frac{\partial}{\partial\theta}\left(\frac{\sigma^2}{\sin\theta}\frac{\partial\tilde U}{\partial\theta}\right)+\frac{1}{R_0\sigma\sin\theta}\frac{\partial^2\tilde U}{\partial\varphi^2}-\left[1-i\varepsilon\frac{1}{r_0\sigma\sin^2\theta}\right]\frac{\partial^2\tilde T}{\partial\varphi^2}=\tilde f(\theta,\varphi)$$

$$-i\varepsilon\left[\frac{R_0}{r_0^2\sigma}\frac{\partial}{\partial\theta}\left(\frac{\sigma^2}{\sin\theta}\frac{\partial\tilde T}{\partial\theta}\right)+\frac{1}{R_0\sigma\sin\theta}\frac{\partial^2\tilde T}{\partial\varphi^2}\right]+\tilde T+\frac{1}{r_0\sigma\sin^2\theta}\tilde U=\frac{R_0\sigma}{\sin\theta}q_n \qquad (4)$$

where

$$\tilde f(\theta,\varphi)=\frac{1}{r_0R_0\sigma}\left\{\frac{\partial}{\partial\theta}\left[(q_n\cos\theta-q_1\sin\theta)\frac{R_0^3\sigma^3}{\sin\theta}\right]+\frac{\partial q_2}{\partial\varphi}r_0R_0^2\sigma^2\right\}. \qquad (5)$$

Eliminating $\tilde U$ from eqn (4), we obtain

$$\sigma^4\frac{\partial^4\tilde T}{\partial\theta^4}+8\mu\sigma^3\cos\theta\frac{\partial^3\tilde T}{\partial\theta^3}+\left[i\frac{\mu r_0}{\varepsilon}\sigma^3\sin\theta+\sigma^2(1+14\mu^2-4\mu\sin\theta-19\mu^2\sin^2\theta)\right]\frac{\partial^2\tilde T}{\partial\theta^2}$$

$$+\left[i\frac{\mu r_0}{\varepsilon}\sigma^2\cos\theta(3+7\mu\sin\theta)+\mu\sigma\cos\theta(2+4\mu^2-6\mu\sin\theta-12\mu^2\sin^2\theta)\right]\frac{\partial\tilde T}{\partial\theta}$$

$$+\mu^4\frac{\partial^4\tilde T}{\partial\varphi^4}+2\mu^2\sigma^2\frac{\partial^4\tilde T}{\partial\theta^2\partial\varphi^2}+4\mu^3\sigma\cos\theta\frac{\partial^3\tilde T}{\partial\theta\partial\varphi^2}$$

$$+\left[i\frac{\mu^2 r_0}{\varepsilon}\sigma^2+\mu^3(\mu-2\sin\theta-2\mu\sin^2\theta)\right]\frac{\partial^2\tilde T}{\partial\varphi^2}$$

$$+i\frac{\mu r_0}{\varepsilon}\sigma[7\mu+(9\mu^2-2)\sin\theta-12\mu\sin^2\theta-12\mu^2\sin^3\theta]\tilde T=\tilde F(\theta,\varphi) \qquad (6)$$

where

$$\tilde F(\theta,\varphi)=-i\frac{\mu^2r_0}{\varepsilon}\sigma\left\{\tilde f(\theta,\varphi)-\frac{r_0}{\mu^2\sigma}\frac{\partial}{\partial\theta}\left[\frac{\sigma^2}{\sin^2\theta}\frac{\partial}{\partial\theta}(q_n\sigma^2\sin\theta)\right]+\frac{r_0}{\sigma\sin\theta}\frac{\partial^2}{\partial\varphi^2}(q_n\sigma^2\sin\theta)\right\}. \qquad (7)$$

The loadings and all the variables in eqn (6) can be expanded in Fourier series along the circular direction of toroidal shells. We let their mth order harmonic components be

$$(\tilde{T}, q_1, q_n, N_1, N_2, M_1, M_2, u, w)(\theta, \varphi) = (\tilde{T}_m, q_{1m}, q_{nm}, N_{1m}, N_{2m}, M_{1m}, M_{2m}, u_m, w_m)(\theta)\cos m\varphi$$

$$(q_2, S, H, v)(\theta, \varphi) = (q_{2m}, S_m, H_m, v_m)(\theta)\sin m\varphi. \tag{8}$$

Inserting eqn (8) in eqns (6) and (7), we obtain the equation

$$\sigma^4\frac{d^4\tilde{T}_m}{d\theta^4} + 8\mu\sigma^3\cos\theta\frac{d^3\tilde{T}_m}{d\theta^3} + \left[i\frac{\mu r_0}{\varepsilon}\sigma^3\sin\theta + \sigma^2(1+14\mu^2-2\mu^2m^2-4\mu\sin\theta-19\mu^2\sin^2\theta)\right]\frac{d^2\tilde{T}_m}{d\theta^2}$$
$$+\left[i\frac{\mu r_0}{\varepsilon}\sigma^2\cos\theta(3+7\mu\sin\theta) + \mu\sigma\cos\theta(2+4\mu^2-4\mu^2m^2-6\mu\sin\theta-12\mu^2\sin^2\theta)\right]\frac{d\tilde{T}_m}{d\theta}$$
$$+\left\{i\frac{\mu r_0}{\varepsilon}\sigma[7\mu-\mu m^2+(9\mu^2-\mu^2m^2-2)\sin\theta-12\mu\sin^2\theta-12\mu^2\sin^3\theta]\right.$$
$$\left.+[\mu^4m^4-\mu^4m^2+2\mu^3m^2\sin\theta+2\mu^4m^2\sin^2\theta]\right\}\tilde{T}_m = \tilde{F}_m(\theta) \tag{9}$$

where

$$\tilde{F}_m(\theta) = -i\frac{\mu^2 r_0}{\varepsilon}\sigma\left\{\frac{\mu}{r_0^2\sigma}\frac{d}{d\theta}\left(q_{nm}\frac{r_0^3\sigma^3}{\mu^3\sin\theta}\right) - \frac{r_0}{\mu^2\sigma}\frac{d}{d\theta}\left[\frac{\sigma^2}{\sin\theta}\frac{d}{d\theta}(q_{nm}\sigma^2\sin\theta)\right] - m^2 r_0\sigma q_{nm}\right\}. \tag{10}$$

In terms of $\tilde{T}_m$, the mth order harmonic components of stress resultants are expressed as

$$N_{1m} = -\frac{\varepsilon}{r_0}\sigma\,\mathrm{Im}\left(\frac{d^2\tilde{T}_m}{d\theta^2}\right) - \frac{\mu\varepsilon}{r_0}\cos\theta\,\mathrm{Im}\left(\frac{d\tilde{T}_m}{d\theta}\right) - \mu\sin\theta\,\mathrm{Re}(\tilde{T}_m) + \frac{\varepsilon\mu^2m^2}{r_0\sigma}\mathrm{Im}(\tilde{T}_m) + q_{nm}r_0\sigma$$

$$N_{2m} = \frac{\varepsilon}{r_0}\sigma\,\mathrm{Im}\left(\frac{d^2\tilde{T}_m}{d\theta^2}\right) + \frac{\mu\varepsilon}{r_0}\cos\theta\,\mathrm{Im}\left(\frac{d\tilde{T}_m}{d\theta}\right) + \sigma\,\mathrm{Re}(\tilde{T}_m) - \frac{\varepsilon\mu^2m^2}{r_0\sigma}\mathrm{Im}(\tilde{T}_m) - q_{nm}r_0\sigma$$

$$mS_m = \frac{\varepsilon\sigma^2}{\mu r_0}\mathrm{Im}\left(\frac{d^3\tilde{T}_m}{d\theta^3}\right) + 4\frac{\varepsilon\sigma}{r_0}\cos\theta\,\mathrm{Im}\left(\frac{d^2\tilde{T}_m}{d\theta^2}\right) + \sigma\sin\theta\,\mathrm{Re}\left(\frac{d\tilde{T}_m}{d\theta}\right)$$
$$+\frac{\varepsilon}{\mu r_0}(1+2\mu^2-\mu^2m^2-3\mu^2\sin^2\theta)\,\mathrm{Im}\left(\frac{d\tilde{T}_m}{d\theta}\right)$$
$$+\cos\theta(2+3\mu\sin\theta)\,\mathrm{Re}(\tilde{T}_m) - \frac{\varepsilon\mu^2m^2}{r_0\sigma}\cos\theta\,\mathrm{Im}(\tilde{T}_m) - 3q_{nm}r_0\sigma\cos\theta$$

$$M_{1m} = \frac{\varepsilon^2}{r_0}(1-\nu)\sigma\,\mathrm{Re}\left(\frac{d^2\tilde{T}_m}{d\theta^2}\right) + \frac{\mu\varepsilon^2}{r_0}(1-\nu)\cos\theta\,\mathrm{Re}\left(\frac{d\tilde{T}_m}{d\theta}\right)$$
$$-\frac{\varepsilon^2\mu^2m^2}{r_0\sigma}(1-\nu)\,\mathrm{Re}(\tilde{T}_m) - \varepsilon(\sigma-\nu\mu\sin\theta)\,\mathrm{Im}(\tilde{T}_m)$$

$$M_{2m} = -\frac{\varepsilon^2}{r_0}(1-\nu)\sigma\,\mathrm{Re}\left(\frac{d^2\tilde{T}_m}{d\theta^2}\right) - \frac{\mu\varepsilon^2}{r_0}(1-\nu)\cos\theta\,\mathrm{Re}\left(\frac{d\tilde{T}_m}{d\theta}\right)$$
$$+\frac{\varepsilon^2\mu^2m^2}{r_0\sigma}(1-\nu)\,\mathrm{Re}(\tilde{T}_m) + \varepsilon(\mu\sin\theta-\nu\sigma)\,\mathrm{Im}(\tilde{T}_m)$$

$$mH_m = -\frac{\varepsilon^2}{\mu r_0}(1-\nu)\sigma^2\,\mathrm{Re}\left(\frac{\mathrm{d}^3\tilde{T}_m}{\mathrm{d}\theta^3}\right)-4\frac{\varepsilon^2}{r_0}(1-\nu)\sigma\cos\theta\,\mathrm{Re}\left(\frac{\mathrm{d}^2\tilde{T}_m}{\mathrm{d}\theta^2}\right)$$

$$-\frac{\varepsilon^2}{\mu r_0}(1-\nu)(1+2\mu^2-\mu^2m^2-3\mu^2\sin^2\theta)\,\mathrm{Re}\left(\frac{\mathrm{d}\tilde{T}_m}{\mathrm{d}\theta}\right)+\varepsilon(1-\nu)\sigma\sin\theta\,\mathrm{Im}\left(\frac{\mathrm{d}\tilde{T}_m}{\mathrm{d}\theta}\right)$$

$$+\frac{\varepsilon^2\mu^2m^2}{r_0\sigma}(1-\nu)\cos\theta\,\mathrm{Re}\,(\tilde{T}_m)+\varepsilon(1-\nu)\cos\theta(2+3\mu\sin\theta)\,\mathrm{Im}\,(\tilde{T}_m). \quad (11)$$

The mth order harmonic components of displacements u_m and w_m are the respective solutions of the equation

$$\frac{\mathrm{d}^2w_m}{\mathrm{d}\theta^2}+w_m = \frac{r_0}{Eh}(N_{1m}-\nu N_{2m})-\frac{12r_0^2}{Eh^3}(M_{2m}-\nu M_{1m}) \quad (12)$$

and the equation

$$\frac{\mathrm{d}u_m}{\mathrm{d}\theta} = \frac{r_0}{Eh}(N_{1m}-\nu N_{2m})-w_m. \quad (13)$$

The displacement v_m is

$$mv_m = \frac{r_0\sigma}{\mu Eh}\left\{\frac{\varepsilon\sigma}{r_0}(1+\nu)\,\mathrm{Im}\left(\frac{\mathrm{d}^2\tilde{T}_m}{\mathrm{d}\theta^2}\right)+(1+\nu)\frac{\varepsilon\mu}{r_0}\cos\theta\,\mathrm{Im}\left(\frac{\mathrm{d}\tilde{T}_m}{\mathrm{d}\theta}\right)+[1+(1+\nu)\mu\sin\theta]\,\mathrm{Re}\,(\tilde{T}_m)\right.$$

$$\left.-(1+\nu)\frac{\varepsilon\mu^2m^2}{r_0\sigma}\,\mathrm{Im}\,(\tilde{T}_m)-q_{nm}(1+\nu)r_0\sigma\right\}-(u_m\cos\theta+w_m\sin\theta). \quad (14)$$

For the asymptotic solutions of eqn (9) to be valid uniformly in the entire toroid, we define the transformation by

$$z = \left[\frac{3}{2m^2}\int_0^\theta\left(\frac{\sin\theta}{1+\mu\sin\theta}\right)^{1/2}\mathrm{d}\theta\right]^{2/3} \quad (15)$$

and

$$\tilde{T}_m = \frac{m^3}{(1+\mu\sin\theta)^2}\left[\frac{\sin\theta}{(1+\mu\sin\theta)z}\right]^{-3/4}\psi. \quad (16)$$

Thus, eqn (9) is transformed to

$$\frac{\mathrm{d}^4\psi}{\mathrm{d}z^4}+\lambda^2\left(z\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}+L_1\frac{\mathrm{d}\psi}{\mathrm{d}z}+L_2\psi\right)+\left(L_3\frac{\mathrm{d}^2\psi}{\mathrm{d}z^2}+L_4\frac{\mathrm{d}\psi}{\mathrm{d}z}+L_5\psi\right) = \lambda^2G_n+G_\tau \quad (17)$$

in which λ is a large complex-valued parameter, L_1, L_2, L_3, L_4, L_5, G_n and G_τ are the real-valued functions of z and are analytic at the transition point $z = 0$. The nonhomogeneous term G_τ vanishes if the loadings have only normal components. The expressions for these coefficients are in Appendix B. However, it is necessary to indicate that

$$L_1(0) = 3. \tag{18}$$

Equation (17) is the fundamental equation. The present derivation is a modified version of the account given by Xia and Zhang (1986).

3. THE FORM OF THE ASYMPTOTIC EXPANSION

We find first the "local solutions" which are valid only in the vicinity of the transition point $z = 0$. It is evident that they are reduced by letting $z \to 0$ in the solutions which are valid everywhere in the entire toroid. Therefore, they can help us infer the extensive forms of the uniformly valid solutions.

In order to find the local solutions we derive the "local equation" which is valid also only in the vicinity of the transition point by letting

$$\zeta = -\lambda^{2/3} z \tag{19}$$

and by using the constants

$$l_2 = L_2(0),\quad l_3 = L_3(0),\quad l_4 = L_4(0),\quad l_5 = L_5(0),\quad g_n = G_n(0),\quad g_\tau = G_\tau(0) \tag{20}$$

instead of the correspondent coefficients $L_1(z)$, $L_2(z) \ldots G_\tau(z)$, in the form

$$\psi^{\mathrm{IV}} - \zeta\psi'' - 3\psi' + \lambda^{-2/3} l_2 \psi + \lambda^{-4/3} l_3 \psi'' - \lambda^{-6/3} l_4 \psi' = \lambda^{-2/3} g_n \tag{21}$$

where use has been made of eqn (18). Primes indicate differentiation with respect to ζ. The terms of order $O(\lambda^{-8/3})$ are omitted, since, they are beyond the accuracy of the theory of thin shells.

3.1. *Expansions of the homogeneous equations*

Substituting

$$\psi(\zeta, \lambda) = \sum_{n=0}^{\infty} (\lambda^{-2/3})^n \psi_n(\zeta) \tag{22}$$

into the homogeneous part of eqn (21), equating coefficients of like powers of $\lambda^{-2/3}$, we obtain

$$\mathbb{T}\mathbb{D}\psi_0 = 0 \tag{23}$$

$$\mathbb{T}\mathbb{D}\psi_n = -l_2\psi_{n-1} - l_3\psi''_{n-2} + l_4\psi'_{n-3} \quad (n = 1, 2 \ldots) \tag{24}$$

where all coefficients with negative subscripts are defined to be zero. The differential operator $\mathbb{T}$ is of the form

$$\mathbb{T} = \mathbb{D}^3 - \zeta\mathbb{D} - 3,\quad \mathbb{D} = \frac{\mathrm{d}}{\mathrm{d}\zeta}. \tag{25}$$

The fourth order comparison eqn (23) has four homogeneous solutions. As can be seen in Appendix A, they are

$$\psi_0^{(k)} = A_{k+1}(\zeta, -1),\quad \psi_0^{(3)} = B_1(\zeta, -1),\quad \psi_0^{(4)} = B_0(\zeta, 1) = 1 \tag{26}$$

where $k = 1, 2$.

The higher approximation can be found by substituting eqn (26) into eqn (24). For example, substituting $\psi_0^{(k)}$ into eqn (24) of $n = 1$ and comparing with eqn (A6) of $p = -1$, we obtain

$$\psi_1^{(k)} = l_2 A_{k+1}(\zeta, 0). \tag{27}$$

Then, substituting $\psi_0^{(k)}$ and $\psi_1^{(k)}$ into eqn (24) of $n = 2$, noting eqn (A5) and comparing with eqn (A6) of $p = 0$ or $p = -3$, respectively, we obtain

$$\psi_2^{(k)} = \tfrac{1}{2} l_2^2 A_{k+1}(\zeta, 1) - l_3 A_{k+1}(\zeta, -2, 0) \tag{28}$$

where $A_{k+1}(\zeta, -2, 0)$ has been used according to eqn (A24), instead of $A_{k+1}(\zeta, -2)$. Moreover, substituting $\psi_0^{(k)}$, $\psi_1^{(k)}$ and $\psi_2^{(k)}$ into eqn (24) of $n = 3$, noting eqn (A5) and comparing with eqn (A6) of $p = 1$ and eqn (A22) of $p = -2$, $q = 1$, respectively, we obtain

$$\psi_3^{(k)} = \tfrac{1}{6} l_2^3 A_{k+1}(\zeta, 2) + l_4 A_{k+1}(\zeta, -1, 1). \tag{29}$$

Thus, we can obtain $\psi_4^{(k)}$, $\psi_5^{(k)}$ . . . by repeating in the same way. Substituting all of them into eqn (22) and using eqns (A12) and (A23), we conclude that the formal expansions are of the form

$$\psi^{(k)}(\zeta, \lambda) = \alpha A_{k+1}(\zeta, -1) + \lambda^{-2/3} \beta A_{k+1}(\zeta, 0) + \lambda^{-4/3} \gamma A_{k+1}(\zeta, 1) + \lambda^{-2}\{a A_{k+1}(\zeta, -1, 1) + \lambda^{-2/3} b A_{k+1}(\zeta, 0, 1) + \lambda^{-4/3} c A_{k+1}(\zeta, 1, 1)\} + \ldots \tag{30}$$

where the coefficients α, β . . . are the known constants composed of the coefficients in eqn (21) and possess expansions for λ^{-2}, for example

$$\alpha = \alpha(\lambda) = \sum_{n=0}^{\infty} \alpha_n (\lambda^{-2})^n, \quad \beta = \beta(\lambda) = \sum_{n=0}^{\infty} \beta_n (\lambda^{-2})^n. \tag{31}$$

In a similar way, the formal expansion of the third independent solution to eqn (21), whose first approximation is $\psi_0^{(3)}$ shown in eqn (26), is of the form

$$\psi^{(3)}(\zeta, \lambda) = \alpha B_1(\zeta, -1) + \lambda^{-2/3} \beta B_1(\zeta, 0) + \lambda^{-4/3} \gamma B_1(\zeta, 1, 1) + \lambda^{-4/3} \delta + \lambda^{-2}\{a B_1(\zeta, -1, 2) + \lambda^{-2/3} b B_1(\zeta, 0, 2) + \lambda^{-4/3} c B_1(\zeta, 1, 2)\} + \ldots \tag{32}$$

in which the coefficients, except δ, are denoted in terms of the same notations as in eqn (30) because they are equal to each other.

To derive the formal expansion of the fourth independent solution to eqn (21) whose first approximation is $\psi_0^{(4)} = 1$ shown in eqn (26), we begin with the nonhomogeneous equation of the form

$$\mathbb{T}\mathbb{D}\psi_1^{(4)} = -l_2 \tag{33}$$

which is obtained by substituting $\psi_0^{(4)} = 1$ into eqn (24) of $n = 1$.

It is easily seen, by letting $p = 1$ in eqn (A8) and noting eqn (A10), that $-\tfrac{1}{3} B_0(\zeta, 2)$ is a particular solution of the nonhomogeneous equation $\mathbb{T}\mathbb{D}u = 1$. Thus,

$$\psi_1^{(4)} = \tfrac{1}{3} l_2 B_0(\zeta, 2). \tag{34}$$

In a similar way, we can find $\psi_2^{(4)}$, $\psi_3^{(4)}$ . . . which are expressed in terms of $B_0(\zeta, p)$ ($p = 1, 2, 3$ . . .). In addition, we note from eqn (A10) that $B_0(\zeta, p)$ are polynomials in ζ.

Then, by noting eqns (19) and (22) we know that the complete expansion of the fourth solution must be of the form

$$\psi^{(4)} = \psi^{(4)}(z,\lambda) = \sum_{n=0}^{\infty} (\lambda^{-2})^n \psi_n^{(4)}(z) \tag{35}$$

which is a slowly varying function.

3.2. *Expansion of a particular solution*

We assume a particular solution to eqn (21) having the form

$$\psi^*(\zeta,\lambda) = \lambda^{-2/3}\left\{\sum_{n=0}^{\infty} (\lambda^{-2/3})^n \psi_n^*(\zeta)\right\}. \tag{36}$$

Substituting eqn (36) into eqn (21) yields a series of equations in the form

$$\mathbb{T}\mathbb{D}\psi_0^* = g_n \tag{37}$$

$$\mathbb{T}\mathbb{D}\psi_n^* = -l_2\psi_{n-1}^* - l_3\psi_{n-2}^{*\prime\prime} + l_4\psi_{n-3}^{*\prime}. \tag{38}$$

We find from eqns (24), (33) and (35) that the particular solution ψ^* has the same expansion as $\psi^{(4)}$ of the form

$$\psi^* = \psi^*(z,\lambda) = \sum_{n=0}^{\infty} (\lambda^{-2})^n \psi_n^*(z) \tag{39}$$

which is also a slowly varying function. This is totally different than the situation in the case of axisymmetric loadings where the particular solution was described by the Lommel function which is a rapidly varying function [see Clark (1958)].

As mentioned before, all of these solutions are valid only in the vicinity of the transition point. However, we may reasonably think that they are also uniformly valid in the entire toroid if the coefficients in them are the unknown functions of z instead of the known constants. In other words, we will find that such uniformly valid solutions of eqn (17) have the same expressions as eqns (30), (32), (35) and (39), in which, however, the coefficients are the unknown functions of z, but the known constants have also the same expansions of eqn (31).

4. UNIFORMLY VALID EXPANSIONS IN THE ENTIRE TOROID

We only need to determine the coefficients in eqns (30), (32), (35) and (39).

4.1. *The first and second homogeneous solutions*

Substituting eqn (30) into eqn (17), using eqns (A5), (A21), (A12), (A23) and (A24) then equating the coefficients of $A_{k+1}(\zeta,p,q)$ ($p = -1, 0, 1$; $q = 0, 1, 2\ldots$) we obtain the ordinary differential equations in terms of the unknown coefficients in eqn (30). All of these unknown coefficients can be written in the form of the asymptotic series as eqn (31). Substituting them into the equations, we then obtain

$$2z^2\alpha_0' + 3z\alpha_0 - zL_1\alpha_0 = 0 \tag{40}$$

$$2z\beta_0' - (L_1 - 2)\beta_0 = 5z\alpha_0'' + 8\alpha_0' - L_1\alpha_0' - L_2\alpha_0 + zL_3\alpha_0 \tag{41}$$

$$z\gamma_0'' + L_1\gamma_0' + L_2\gamma_0 = 0 \tag{42}$$

$$2z^2a_0'+3za_0-zL_1a_0 = 0. \tag{43}$$

The solution of these equations yields the complete asymptotic expansions within the accuracy of the theory of thin shells in the form

$$\psi^{(k)}(\zeta,\lambda) = \alpha_0 A_{k+1}(\zeta,-1)+\lambda^{-2/3}\beta_0 A_{k+1}(\zeta,0)+\lambda^{-4/3}\gamma_0 A_{k+1}(\zeta,1)+\lambda^{-2}a_0 A_{k+1}(\zeta,-1,1), \tag{44}$$

where

$$\alpha_0(z) = a_0(z) = z^{-3/2}\exp\left\{\frac{1}{2}\int_0^z \frac{L_1(\tau)}{\tau}\,\mathrm{d}\tau\right\} \tag{45}$$

$$\beta_0(z) = \frac{1}{8}z^{-1}\exp\left\{\frac{1}{2}\int_0^z \frac{L_1(\tau)}{\tau}\,\mathrm{d}\tau\right\}\int_0^z M(\tau)\,\mathrm{d}\tau \tag{46}$$

$$M(\tau) = \tau^{-5/2}[27-18L_1(\tau)+3L_1^2(\tau)]+2\tau^{-3/2}[5L_1'(\tau)-2L_2(\tau)]+4L_3(\tau). \tag{47}$$

It is argued that $\alpha_0(z)$ and $\beta_0(z)$ are analytic at the transition point $z = 0$. In addition, eqn (42) is the homogeneous counterpart of the membrane equation

$$z\phi''+L_1\phi'+L_2\phi = G_n \tag{48}$$

which is obtained by putting $\lambda\to\infty$ in the fundamental eqn (17). It can be shown that only one homogeneous solution of eqn (48), specified by $\phi_1^{(0)}(z)$, is analytic at the transition point $z = 0$ and the other independent homogeneous solution, specified by $\phi_2^{(0)}(z)$, is singular at $z = 0$. Moreover, there exists such a particular solution of eqn (48), specified by $\phi_*^{(0)}(z)$, which is analytic at $z = 0$. We let

$$\gamma_0(z) = \phi_1^{(0)}(z). \tag{49}$$

4.2. *The third homogeneous solution*

Similarly, substituting eqn (32) into eqn (17) we obtain the complete expansion of the third homogeneous solution within the accuracy of the theory of thin shells in the form

$$\psi^{(3)}(\zeta,\lambda) = \alpha_0 B_1(\zeta,-1)+\lambda^{-2/3}\beta_0 B_1(\zeta,0)+\lambda^{-4/3}\gamma_0 B_1(\zeta,1,1)+\lambda^{-4/3}\delta_0(z). \tag{50}$$

It is remarkable that the last term in eqn (50) is a slowly varying function. α_0, β_0 and γ_0 remain to be determined by eqn (45), (46) and (49), respectively. The slowly varying function δ_0 is a solution of the nonhomogeneous equation

$$z\delta_0''+L_1\delta_0'+L_2\delta_0 = \frac{x(z)}{z} \tag{51}$$

where

$$x(z) = -6\alpha_0''-L_3\alpha_0-z\beta_0''+4\beta_0'-L_1\beta_0'-L_2\beta_0+2z\gamma_0'+L_1\gamma_0-\gamma_0. \tag{52}$$

Thus,

$$\delta_0 = \delta_0(z) = \int_0^z \frac{\phi_1^{(0)}(\tau)\phi_2^{(0)}(z)-\phi_1^{(0)}(z)\phi_2^{(0)}(\tau)}{\phi_1^{(0)}(\tau)\phi_2^{(0)\prime}(\tau)-\phi_1^{(0)\prime}(\tau)\phi_2^{(0)}(\tau)}\frac{x(\tau)}{\tau}\,\mathrm{d}\tau. \tag{53}$$

4.3. *The fourth homogeneous solution and the particular solution*

It is sufficient to take the leading terms in the expansions (35) and (39) in powers of λ^{-2}. Substituting eqns (35) and (39) into eqn (17), we find that the leading terms ψ_0^* and $\psi_0^{(4)}$ satisfy the membrane eqn (48) and its homogeneous counterpart, respectively. Therefore, we let

$$\psi^{(4)} \approx \psi_0^{(4)} = \phi_1^{(0)}(z) \tag{54}$$

and

$$\psi^* \approx \psi_0 = \phi_*^{(0)}(z). \tag{55}$$

All of them are slowly varying functions.

5. THE FINAL RESULTS

A general solution of the fundamental eqn (17) is of the form

$$\psi(\zeta,\lambda) = C_1\psi^{(1)}+C_2\psi^{(2)}+C_3\psi^{(3)}+C_4\psi^{(4)}+\psi^* \tag{56}$$

where the homogeneous solutions $\psi^{(1)}$, $\psi^{(2)}$, $\psi^{(3)}$ and $\psi^{(4)}$ and the particular solution ψ^* are shown in eqns (44), (50), (54) and (55), respectively. Four complex constants C_i will be determined by eight boundary conditions.

With the knowledge of the general solution, the complex variable $\tilde{T}_m$ will follow from eqn (16). Then the stress resultants and displacements can be determined using eqns (11)–(14).

6. COMPARISON

Let $m = 1$ in eqn (8), the loading becomes

$$q_1 = q_{11}(\theta)\cos\varphi, \quad q_2 = q_{21}(\theta)\sin\varphi, \quad q_n = q_{n1}(\theta)\cos\varphi, \tag{57}$$

which is referred to as "wind-type" loading.

Novozhilov (1951) in his monograph has given the asymptotic expressions of the force and moment resultants and the strains for the general shell of revolution subjected to the wind-type loading based on the limit that the special circumferences $\theta = 0$ and $\theta = \pi$ are not on the shell. Obviously, his expressions are still valid for a segment of toroidal shell with positive curvature between and away from $\theta = 0$ and $\theta = \pi$.

In the other respects, the results in the present paper are adaptable for the toroidal shell with any nonsymmetric loadings. Their special case of $m = 1$ and $\theta \neq 0$ and $\theta \neq \pi$ of course must coincide with Novozhilov's results.

Novozhilov's results are only first approximation. All the small quantities higher than order $O(\sqrt{\varepsilon}) \sim O(\sqrt{h})$ in them are omitted. With the same accuracy formula (56) becomes

$$\psi(\zeta,\lambda) = \alpha_0(z)[C_1A_2(\zeta,-1)+C_2A_3(\zeta,-1)]+C_4\phi_1^{(0)}(z)+\phi_*^{(0)}(z) \tag{58}$$

in which, according to eqns (A14)–(A16), A_2 and A_3 are expressed as

$$A_2(\zeta,-1) \approx \tilde{f}(\theta,\varepsilon)[i e^{(1-i)\omega}+e^{-(1-i)\omega}]$$
$$A_3(\zeta,-1) = \tilde{f}(\theta,\varepsilon)[-i e^{(1-i)\omega}+e^{-(1-i)\omega}] \tag{59}$$

where

$$\tilde{f}(\theta,\varepsilon) = \varepsilon^{-1/12}(-i\mu r_0)^{1/12} z^{1/4} \tag{60}$$

$$\omega = \frac{1}{\sqrt{2}}\int_{\theta_0}^{\theta}\sqrt{\frac{\sin\theta}{R_0\sigma\varepsilon}}\, r_0\, d\theta \tag{61}$$

and $\theta_0 \neq 0$ is the upper boundary circumference.

Substituting eqns (59)–(61) into eqns (58) and (16) yields

$$\tilde{T}_1 = \left\{\frac{1}{\sigma^2}\left(\frac{\sin\theta}{\sigma z}\right)^{-3/4}\tilde{f}(\theta,\varepsilon)\alpha_0(z)[\tilde{C}_1 e^{-(1-i)\omega}+\tilde{C}_2 e^{(1-i)\omega}]\right\} + \frac{1}{\sigma^2}\left(\frac{\sin\theta}{\sigma z}\right)^{-3/4}[C_4\phi_1^{(0)}(z)+\phi_*^{(0)}(z)] \tag{62}$$

where

$$\tilde{C}_1 = C_1 + C_2, \quad \tilde{C}_2 = i(C_1 - C_2). \tag{63}$$

In the first braces of eqn (62), the coefficient $1/\sigma^2(\sin\theta/\sigma z)^{-3/4}\tilde{f}(\theta,\varepsilon)\alpha_0(z)$ has to be seen as a constant, since only the primary term is retained in the derivative of $\tilde{T}_1$ with respect to θ. It can be absorbed by the arbitrary complex constants $\tilde{C}_1$ and $\tilde{C}_2$. Thus, eqn (62) is rewritten as

$$\tilde{T}_1 = \tilde{\tau} + T^{(0)} \tag{64}$$

in which

$$\tilde{\tau} = \tilde{C}_1 e^{-(1-i)\omega} + \tilde{C}_2 e^{(1-i)\omega} \tag{65}$$

and

$$T^{(0)} = \frac{1}{\sigma^2}\left(\frac{\sin\theta}{\sigma z}\right)^{-3/4}[C_4\phi_1^{(0)}(z)+\phi_*^{(0)}(z)]. \tag{66}$$

As it can be seen that eqn (65) is the same as Novozhilov's formula (4.13.24) and that eqn (66) must satisfy the membrane eqn (9) or the membrane eqn (1) replaced with eqns (2) and (8) and $m = 1$. The reason of the latter is because $C_4\phi_1^{(0)} + \phi_*^{(0)}$ satisfies the membrane eqn (48) and because of eqn (16).

All the stress resultants and stress couples can be obtained by substituting eqn (64) into eqn (11). As an example we consider

$$N_{11} = -\frac{\mu\varepsilon}{r_0}\cos\theta\,\mathrm{Im}\left(\frac{d\tilde{\tau}}{d\theta}\right) + \left\{-\frac{\varepsilon\sigma}{r_0}\mathrm{Im}\left(\frac{d^2\tilde{\tau}}{d\theta^2}\right) - \mu\sin\theta\,\mathrm{Re}\,(\tilde{\tau}) - \mu\sin\theta\,\mathrm{Re}\,(T^{(0)}) + q_{n1}r_0\sigma\right\} \tag{67}$$

where

$$-\frac{\mu\varepsilon}{r_0}\cos\theta\,\mathrm{Im}\left(\frac{\mathrm{d}\tilde{\tau}}{\mathrm{d}\theta}\right) = -\frac{1}{\sqrt{2}}\sqrt{\frac{\varepsilon\sin\theta}{R_0\sigma}}\cos\theta\,\{[(C_1'-C_1'')\cos\omega-(C_1'+C_1'')\sin\omega]\,\mathrm{e}^{-\omega} - [(C_2'-C_2'')\cos\omega+(C_2'+C_2'')\sin\omega]\,\mathrm{e}^{\omega}\}. \quad (68)$$

It can be seen that eqn (68) is identical to Novozhilov's formula (3.14.9) if $\cos\theta$ is replaced with $\cot\theta$. However, both $\cos\theta$ and $\cot\theta$ can be absorbed in the arbitrary constants C_1', C_1'', C_2' and C_2'' as mentioned above. Thus, we can see eqn (68) as the same as Novozhilov's resultant.

In addition, the second summand of eqn (67) is equal to the membrane stress resultant $N_{11}^{(0)}$. The real part of the complex constant C_4 in $N_{11}^{(0)}$ [see eqn (66)] can be determined by giving the boundary membrane stress resultant

$$N_{11}^* = N_{11}^{(0)}|_{\theta=\theta_0} = \left\{-\frac{\varepsilon\sigma}{r_0}\mathrm{Im}\left(\frac{\mathrm{d}^2\tilde{\tau}}{\mathrm{d}\theta^2}\right)-\mu\sin\theta\,\mathrm{Re}\,(\tilde{\tau})-\mu\sin\theta\,\mathrm{Re}\,(T^{(0)})+q_{n1}r_0\sigma\right\}\Bigg|_{\theta=\theta_0}. \quad (69)$$

As for the determination of the imaginary part of C_4, we have to give the boundary shear stress resultant S^*. This is the same as Novozhilov's approach.

Finally, we have

$$N_{11} = -\frac{1}{\sqrt{2}}\sqrt{\frac{\varepsilon\sin\theta}{R_0\sigma}}\cos\theta\{[(C_1'-C_1'')\cos\omega-(C_1'+C_1'')\sin\omega]\,\mathrm{e}^{-\omega} - [(C_2'-C_2'')\cos\omega+(C_2'+C_2'')\sin\omega]\,\mathrm{e}^{\omega}\}+N_{11}^{(0)} \quad (70)$$

which is identical to Novozhilov's formulae (3.14.9) or (3.14.22).

Note that there are only four constants C_1', C_1'', C_2' and C_2'' to be determined in the present case instead of eight constants in the other nonsymmetric cases. This is a special feature of the wind-type loading on which Novozhilov has made a detailed explanation.

7. CONCLUSIONS

(1) The fact that the particular solution $\psi^*(z)$ satisfies the membrane eqn (48) shows that nonaxisymmetric loadings are equilibrated totally by membrane stress resultants. This differs completely from the axisymmetric loading situation in which the axisymmetric surface loadings are equilibrated by both the membrane stress resultants and the bending moment.

(2) The existence of the third homogeneous solution $\psi^{(3)}$ shows that bending moments exist everywhere in the entire shell and their existence is independent from the loadings but dependent on the boundary conditions. This is completely different from the axisymmetric loading situation in which the bending moments existing in the entire toroid are induced by the loadings.

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APPENDIX A

Drazin and Reid (1981) have shown that the generalized Airy functions

$$A_k(\zeta,p) = \frac{1}{2\pi i}\int_{I_k} t^{-p}\exp(\zeta t-\tfrac{1}{3}t^3)\,dt \quad (k = 1,2,3;\ p = 0,\pm 1,\pm 2\ldots) \tag{A1}$$

$$B_k(\zeta,p) = \int_{I_k} t^{-p}\exp(\zeta t-\tfrac{1}{3}t^3)\,dt \quad (k = 1,2,3;\ p = 0,-1,-2,\ldots) \tag{A2}$$

and

$$B_0(\zeta,p) = \frac{1}{2\pi i}\int_{I_0} t^{-p}\exp(\zeta t-\tfrac{1}{3}t^3)\,dt \quad (p = 0,\pm 1,\pm 2\ldots) \tag{A3}$$

are solutions of the differential equation

$$(\mathbb{D}^3-\zeta\mathbb{D}+p-1)u = 0 \tag{A4}$$

where the contours are shown in Fig. A1. The derivatives of these solutions satisfy the relation

$$\mathbb{D}^n A_k(\zeta,p) = A_k(\zeta,p-n),\quad \mathbb{D}^n B_k(\zeta,p) = B_k(\zeta,p-n),\quad \mathbb{D}^n B_0(\zeta,p) = B_0(\zeta,p-n). \tag{A5}$$

It is not difficult from eqn (A4) to arrive at the following relations of identical form

$$\mathbb{T}\mathbb{D}A_k(\zeta,p+1) = -(p+2)A_k(\zeta,p) \tag{A6}$$

$$\mathbb{T}\mathbb{D}B_k(\zeta,p+1) = -(p+2)B_k(\zeta,p) \tag{A7}$$

and

$$\mathbb{T}\mathbb{D}B_0(\zeta,p+1) = -(p+2)B_0(\zeta,p) \tag{A8}$$

where

$$\mathbb{T} = \mathbb{D}^3-\zeta\mathbb{D}-3. \tag{A9}$$

Note from eqn (A3) that $B_0(\zeta,p) \equiv 0$, if $p \leqslant 0$; otherwise, it is a polynomial in ζ of degree $p-1$ which, by the residue theorem, is simply the coefficient of t^{p-1} in the expansion of $\exp(\zeta t-\frac{1}{3}t^3)$. The first few terms of these polynomials are

$$\begin{aligned} B_0(\zeta,1) &= 1 & B_0(\zeta,4) &= \frac{1}{3!}\zeta^3-\frac{1}{3} \\ B_0(\zeta,2) &= \zeta & B_0(\zeta,5) &= \frac{1}{4!}\zeta^4-\frac{1}{3}\zeta \\ B_0(\zeta,3) &= \frac{1}{2!}\zeta^2 & B_0(\zeta,6) &= \frac{1}{5!}\zeta^5-\frac{1}{6}\zeta^2. \end{aligned} \tag{A10}$$

It is seen by putting $p = -2$ in eqns (A6) and (A7) and $p = 0$ in eqn (A8), that $A_k(\zeta,-1)$, $B_k(\zeta,-1)$ and $B_0(\zeta,1)$ $(k = 1, 2, 3)$ are non-trivial solutions of the homogeneous equation $\mathbb{T}\mathbb{D}u = 0$. Moreover, we indicate that

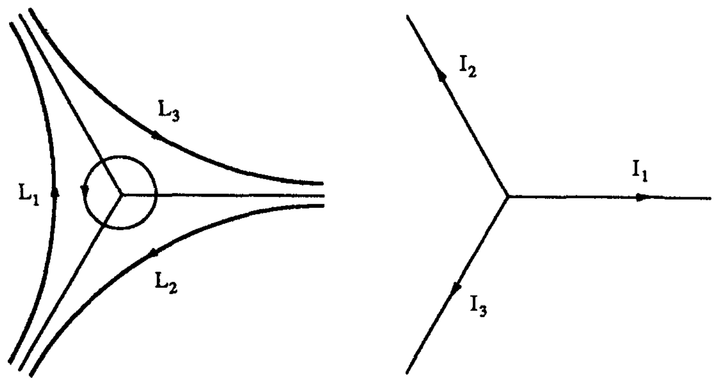

Fig. A1. The paths of integration in the t-plane.

$A_2(\zeta,-1)$, $A_3(\zeta,-1)$, $B_1(\zeta,-1)$ and $B_0(\zeta,1)\equiv 1$ are a linearly independent set of solutions, because their Wronskian is a constant and is given by

$$\mathscr{W}(A_2,A_3,B_1,1)(\zeta,-1) = -\frac{i}{\pi}. \tag{A11}$$

By using eqn (A6) we can immediately obtain the particular integrals of the inhomogeneous equation $\mathbf{T}\mathbf{D}u = A_k(\zeta,p)$ for all values of p except $p=-2$, and have the recursion formula

$$A_k(\zeta,p-3)-\zeta A_k(\zeta,p-1)+(p-1)A_k(\zeta,p) = 0. \tag{A12}$$

Thus, for other values of p, $A_k(\zeta,p)$ can be expressed as a linear combination of $A_k(\zeta,0)$ and $A_k(\zeta,\pm 1)$ with polynomial coefficients.

The asymptotic behaviour of $A_k(\zeta,p)$ is given by

$$\begin{aligned} &A_1(\zeta,p)\sim A_-(\zeta,p) \quad (\zeta\in T_2\cup T_3)\\ &A_2(\zeta,p)\sim iA_+(\zeta,p) \quad (\zeta\in T_3\cup T_1)\\ &A_3(\zeta,p)\sim\begin{cases}-A_-(\zeta,p) & (\zeta\in T_1)\\ -iA_+(\zeta,p) & (\zeta\in T_2)\end{cases}\end{aligned} \tag{A13}$$

and

$$\begin{bmatrix}A_1(\zeta,p)\\A_2(\zeta,p)\\A_3(\zeta,p)\end{bmatrix}\sim\begin{bmatrix}-i & -1 & -1\\ i & -1 & -1\\ -i & -1 & -1\end{bmatrix}\begin{bmatrix}A_+(\zeta,p)\\B_0(\zeta,p)\\A_-(\zeta,p)\end{bmatrix}\ (\zeta\in T_k, p\in\mathbb{Z}) \tag{A14}$$

where T_k are sectors shown in Fig. A2;

$$\begin{aligned} A_\pm(\zeta,p) &= \frac{1}{2\sqrt{\pi}}(\pm 1)^p\zeta^{-(2p+1)/4}\,\mathrm{e}^{\pm\xi}\sum_{S=0}^{\infty}(\pm 1)^S a_S(p)\zeta^{-S}\\ \xi &= \tfrac{2}{3}\zeta^{3/2}\end{aligned} \tag{A15}$$

and

$$\begin{aligned} a_0(p) &= 1,\\ a_1(p) &= \frac{1}{2^3 3^2}(12p^2+24p+5)\\ a_2(p) &= \frac{1}{2^7 3^4}(144p^4+1344p^3+3864p^2+3504p+385)+\dots. \end{aligned} \tag{A16}$$

The functions $B_k(\zeta,p)$ also satisfy eqn (A12) with $A_k(\zeta,p)$ replaced by $B_k(\zeta,p)$, i.e.

$$B_k(\zeta,p-3)-\zeta B_k(\zeta,p-1)+(p-1)B_k(\zeta,p) = 0 \quad (p\leqslant 0). \tag{A17}$$

The other important recursion formula is

$$B_k(\zeta,-2)-\zeta B_k(\zeta,0)-1 = 0. \tag{A18}$$

The asymptotic behaviour of $B_k(\zeta,p)$ in T_k is given by

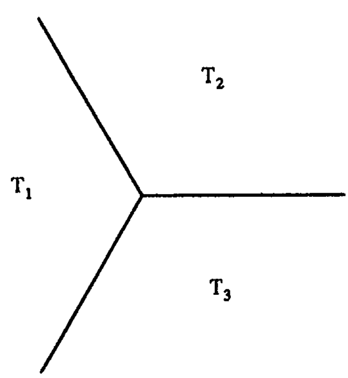

Fig. A2. The sectors for the Airy functions in the ζ-plane.

$$B_k(\zeta,p) \sim (-1)^{1-p}(-p)!\,\zeta^{p-1}\{1-\tfrac{1}{3}(p-1)(p-2)\zeta^{-3}+\ldots\}. \tag{A19}$$

To obtain the particular integrals of the nonhomogeneous equation $\mathbb{T}\mathbb{D}u = A_k(\zeta, -2)$, we consider the third generalized Airy functions defined by Drazin and Reid (1981) as follows:

$$A_k(\zeta,p,q) = \frac{1}{2\pi i}\int_{l_k} t^{-p}(\ln t)^q \exp(\zeta t - \tfrac{1}{3}t^3)\,\mathrm{d}t \tag{A20}$$

where $k = 1, 2, 3$; $P = 0, \pm 1, 2\ldots$; $q = 0, 1, 2\ldots$; and a branch cut has been placed along the positive real axis in the t-plane so that $0 \leqslant pht < 2\pi$. Similarly, the derivatives satisfy the relation

$$\mathbb{D}^n A_k(\zeta,p,q) = A_k(\zeta,p-n,q). \tag{A21}$$

The function $A_k(\zeta,p,q)$ are solutions of the nonhomogeneous equation

$$\mathbb{T}\mathbb{D}A_k(\zeta,p+1,q) = -(p+2)A_k(\zeta,p,q)+qA_k(\zeta,p,q-1) \tag{A22}$$

from which we have the recursion formula

$$A_k(\zeta,p-3,q)-\zeta A_k(\zeta,p-1,q)+(p-1)A_k(\zeta,p,q) = qA_k(\zeta,p,q-1). \tag{A23}$$

It is easily seen that

$$A_k(\zeta,p,0) \equiv A_k(\zeta,p). \tag{A24}$$

The asymptotic expansion of $A_1(\zeta,p,1)$ is

$$A_1(\zeta,p,1) \sim \frac{1}{2\sqrt{\pi}}(-1)^p\zeta^{-(2p+1)/4}\,\mathrm{e}^{-\xi}\sum_{S=0}^{\infty}(-1)^S[(\tfrac{1}{2}\ln\zeta+\pi i)a_S(p)-a'_S(p)]\xi^{-S} \quad (\zeta\in T_2\cup T_3). \tag{A25}$$

The corresponding expansions for $A_2(\zeta,p,1)$ and $A_3(\zeta,p,1)$ then follow from the recursion formula

$$A_2(\zeta,p,1) = \mathrm{e}^{-2(p-1)\pi i/3}[A_1(\zeta\mathrm{e}^{2\pi i/3},p,1)+\tfrac{2}{3}\pi i A_1(\zeta\mathrm{e}^{2\pi i/3},p)]$$
$$A_3(\zeta,p,1) = \mathrm{e}^{2(p-1)\pi i/3}[A_1(\zeta\mathrm{e}^{-2\pi i/3},p,1)-\tfrac{2}{3}\pi i A_1(\zeta\mathrm{e}^{-2\pi i/3},p)]. \tag{A26}$$

To deal with the third homogeneous solution we need to use the functions defined by Drazin and Reid (1981) as follows:

$$B_k(\zeta,p,q) = \frac{1}{2\pi i}\int_{\infty\cdot\exp[2(k-1)\pi i/3]}^{(0_+)} t^{-p}(\ln t)^q\exp(\zeta t-\tfrac{1}{3}t^2)\,\mathrm{d}t \tag{A27}$$

where $k = 1, 2, 3$; $p = 0, \pm 1, \pm 2\ldots$; $q = 0, 1, 2\ldots$. We may note that

$$B_k(\zeta,p,0) \equiv B_0(\zeta,p) \quad (p = 0, \pm 1, \pm 2\ldots) \tag{A28}$$

and

$$B_k(\zeta,p,1) \equiv B_k(\zeta,p) \quad (p = 0, -1, -2\ldots) \tag{A29}$$

for all values of k. The functions $B_k(\zeta,p,q)$ also satisfy eqns (A21)–(A23) with $A_k(\zeta,p,q)$ replaced by $B_k(\xi,p,q)$. The asymptotic expansions of $B_1(\xi,p,q)$ in the sector T_1 are

$$B_1(\zeta,1,1) \sim -\ln\zeta-\gamma+\sum_{n=1}^{\infty}\frac{(3n-1)!}{3^n n!}\zeta^{-3n} \tag{A30}$$

where γ is the Euler constant.

APPENDIX B

The coefficients in eqn (17) are as follows:

$$L_1 = \frac{1}{m^4\sigma}\frac{1}{\phi_0 z'^4}[(\phi_0 z''+2\phi'_0 z')\sin\theta+3\phi_0 z'\cos\theta] \tag{B1}$$

$$L_2 = \frac{1}{m^4\sigma}\frac{1}{z'^4}\left\{\frac{1}{\phi_0}(\phi''_0\sin\theta+3\phi'_0\cos\theta)+\frac{1}{\sigma^2}[\mu(1-m^2)+(\mu^2-\mu^2m^2-2)\sin\theta-4\mu\sin^2\theta-2\mu^2\sin^3\theta]\right\} \tag{B2}$$

$$L_3 = \frac{1}{\sigma^2}\frac{1}{z'^2}(1+2\mu^2-\mu^2m^2+8\mu\sin\theta+5\mu^2\sin^2\theta)+\frac{1}{\phi_0 z'^4}(4\phi_0 z'z'''+3\phi_0 z''^2+12\phi_0' z'z''+6\phi_0'' z'^2) \quad \text{(B3)}$$

$$L_4 = \frac{1}{\phi_0 z'^4}\Big[(\phi_0 z^{(4)}+4\phi_0' z'''+6\phi_0'' z''+4\phi_0''' z') + \frac{1}{\sigma^2}(\phi_0 z''+2\phi_0' z')(1+2\mu^2-2\mu^2m^2+8\mu\sin\theta+5\mu^2\sin^2\theta)\Big]+\frac{\mu\cos\theta}{\sigma^3 z'^3}(6-4\mu^2+4\mu^2m^2+2\mu\sin\theta) \quad \text{(B4)}$$

$$L_5 = \frac{\phi_0^{(4)}}{\phi_0 z'^4}+\frac{\phi_0''}{\sigma^2\phi_0 z'^4}(1+2\mu^2-2\mu^2m^2+8\mu\sin\theta+5\mu^2\sin^2\theta)+\frac{\phi_0'\mu\cos\theta}{\sigma^3\phi_0 z'^4}(6-4\mu^2+4\mu^2m^2+2\mu\sin\theta) - \frac{\mu^2}{\sigma^4 z'^4}[(6-4\mu^2-\mu^2m^4+5\mu^2m^2)+2\mu m^2\sin\theta-(12-2\mu^2+2\mu^2m^2)\sin^2\theta-12\mu\sin^3\theta-4\mu^2\sin^4\theta] \quad \text{(B5)}$$

and

$$G_n = -\frac{r_0 m}{\mu\sigma^2}\left(\frac{z\sigma}{\sin\theta}\right)^{5/4} q_{nm}\Big\{\mu^2m^2\left(1+\frac{\mu}{2}\right)+[(2m^2+\tfrac{9}{4})\mu^3+3\mu]\sin\theta - [\tfrac{1}{2}(m^2+3)\mu^4+9\mu^2]\cos 2\theta-\tfrac{27}{4}\mu^3\sin 3\theta+\tfrac{3}{2}\mu^4\sin 4\theta\Big\} \quad \text{(B6)}$$

where

$$\phi_0 = m^3\left(\frac{\sin\theta}{\sigma z}\right)^{-3/4} \quad \text{(B7)}$$

and

$$(\ )' = \frac{\mathrm{d}}{\mathrm{d}\theta}(\ ).$$

The expression of G_τ is omitted.